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KÄHLERIAN TORSE-FORMING VECTOR FIELDS AND KÄHLERIAN SUBMERSIONS

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Abstract. Let M be a Kählerian manifold and ∇ the Levi-Civita connection of M . In this paper, we consider a linear connection ∇^0 having a certain relation to ∇ and a Kählerian torse-forming vector field on M . The properties of the curvature tensor R^0 of ∇^0 and the Bochner curvature tensors are studied. Also we apply these properties to a Kählerian submersion.

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§1. Introduction

Let M be a real $2m$ -dimensional Kählerian manifold with the complex structure J . We denote by ∇ the Levi-Civita connection of M and by $\mathfrak{X}(M)$ the set of all smooth vector fields on M . In [5], S. Yamaguchi introduced the notion of a Kählerian torse-forming vector field on a Kählerian manifold. If, for any $E \in \mathfrak{X}(M)$, a vector field ξ satisfies

$$(1.1) \quad \nabla_E \xi = aE + bJE + \alpha(E)\xi + \beta(E)J\xi,$$

where a and b are functions on M and α and β are 1-forms on M , then we call such a vector field ξ a Kählerian torse-forming vector field. Moreover, if the associated functions a and b satisfy $a^2 + b^2 > 0$ in M , then we call ξ a proper Kählerian torse-forming vector field.

In this paper, we consider the following linear connection ∇^0 :

$$(1.2) \quad \begin{aligned} \nabla_E^0 F := & \nabla_E F - \rho(E)F - \rho(F)E + \rho(JE)JF + \rho(JF)JE \\ & - f(E, F)\xi + f(JE, F)J\xi \end{aligned}$$

for any $E, F \in \mathfrak{X}(M)$, where ξ is a Kählerian torse-forming vector field, ρ a 1-form on M and f a $(0, 2)$ -tensor field of M respectively. In [9], S. Yamaguchi and W.N. Yu assumed that there exists a local coordinate system $\{x^h\}$ satisfying

$$\nabla_{\partial_i}^0 \partial_j = 0$$

for $1 \leq i, j \leq 2m$ about each point of M , where $\partial_i = \partial/\partial x^i$. They obtained some results on the Bochner curvature tensor, the Ricci tensor, etc. The purpose of this paper is to generalize these results. In §2, we have a relation between the curvature tensor R^0 of ∇^0 and the curvature tensor R of ∇ . Moreover, a relation between the Bochner curvature tensor B of ∇ and the Bochner curvature B^0 with respect to ∇^0 is given. In §3, we apply these relations in §2 to a Kählerian submersion.

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§2. A Kählerian torse-forming vector field on Kählerian manifold

Let (M, g, J) be a real $2m$ -dimensional Kählerian manifold with the complex structure J and Kählerian metric g . For simplicity, we denote the metric g by (\cdot, \cdot) . We put $|X| := \sqrt{(X, X)}$ for $X \in TM$, where TM is the tangent bundle of M . Hereafter, we assume that ξ is a Kählerian torse-forming vector field satisfying (1.1). Let ρ be a 1-form on M and f a $(0, 2)$ -tensor field on M satisfying

$$f(E, F) = f(F, E) \quad \text{and} \quad f(E, JF) = f(F, JE)$$

for any $E, F \in TM$. We define a linear connection ∇^0 by (1.2). Then we can easily obtain

Lemma 2.1. ∇^0 is a torsion free connection and $\nabla^0 J = 0$.

The curvature tensor field R^0 and R are defined by

$$\begin{aligned} R^0(E, F)G &:= \nabla_E^0 \nabla_F^0 G - \nabla_F^0 \nabla_E^0 G - \nabla_{[E, F]}^0 G, \\ R(E, F)G &:= \nabla_E \nabla_F G - \nabla_F \nabla_E G - \nabla_{[E, F]} G \end{aligned}$$

for any $E, F, G \in \mathfrak{X}(M)$ respectively. Using (1.1) and (1.2), by a straightforward but rather complicated computations, we have

$$\begin{aligned} (2.1) \quad R^0(E, F)G - R(E, F)G &= -\{\mu(E, F) - \mu(F, E)\}G - \mu(E, G)F \\ &\quad + \mu(F, G)E + \mu(E, JG)JF - \mu(F, JG)JE \\ &\quad + \{\mu(E, JF) - \mu(F, JE)\}JG - \nu(E, F, G)\xi \\ &\quad + \nu(E, F, JG)J\xi, \end{aligned}$$

where

$$(2.2) \quad \begin{aligned} \mu(E, F) &:= (\nabla_E \rho)(F) + \rho(E)\rho(F) - \rho(JE)\rho(JF) \\ &\quad + \{\rho(\xi) - a\}f(E, F) - \{\rho(J\xi) + b\}f(E, JF), \end{aligned}$$

$$(2.3) \quad \begin{aligned} \nu(E, F, G) &:= f(F, G)\{\alpha(E) - f(E, \xi)\} - f(E, G)\{\alpha(F) - f(F, \xi)\} \\ &\quad + f(JF, G)\{\beta(E) + f(JE, \xi)\} + (\nabla_E f)(F, G) \\ &\quad - f(JE, G)\{\beta(F) + f(JF, \xi)\} - (\nabla_F f)(E, G). \end{aligned}$$

From (2.3), we see that

$$(2.4) \quad \nu(E, F, G) + \nu(F, G, E) + \nu(G, E, F) = 0,$$

$$(2.5) \quad \nu(E, F, JG) + \nu(F, G, JE) + \nu(G, E, JF) = 0.$$

Hereafter, we assume the following equation:

$$(2.6) \quad (R^0(E, F)G, H) + (R^0(E, F)H, G) = 0$$

for every $E, F, G, H \in TM$. Then, from (2.1), it is equivalent to

$$(2.7) \quad \begin{aligned} &-2\{\mu(E, F) - \mu(F, E)\}(G, H) \\ &- \mu(E, G)(F, H) + \mu(F, G)(E, H) - \mu(E, H)(F, G) \\ &+ \mu(F, H)(E, G) + \mu(E, JG)(JF, H) - \mu(F, JG)(JE, H) \\ &+ \mu(E, JH)(JF, G) - \mu(F, JH)(JE, G) \\ &- \nu(E, F, G)(\xi, H) + \nu(E, F, JG)(J\xi, H) \\ &- \nu(E, F, H)(\xi, G) + \nu(E, F, JH)(J\xi, G) = 0. \end{aligned}$$

It can be proved from (2.7) that

$$(2.8) \quad (m+1)\{\mu(E, F) - \mu(F, E)\} + \nu(E, F, \xi) = 0$$

for any $E, F \in TM$. Now we prove

Lemma 2.2. *If ξ is everywhere non-zero and $\dim M = 2m \geq 6$, then we get*

$$(2.9) \quad \mu(E, F) = \mu(F, E),$$

$$(2.10) \quad \nu(E, F, \xi) = 0,$$

$$(2.11) \quad \nu(\xi, E, F) = \nu(\xi, F, E)$$

for every $E, F \in TM$.

Proof. For $p \in M$, $\text{Span}\{\xi_p, J\xi_p\}$ denotes the 2-dimensional subspace spanned by ξ_p and $J\xi_p$. We take two vectors $Y, Z \in (\text{Span}\{\xi_p, J\xi_p\})^\perp$ such that

$$(Y, Y) = (Z, Z) = 1, \quad (Y, Z) = (JY, Z) = 0,$$

where $(\text{Span}\{\xi_p, J\xi_p\})^\perp$ means the orthogonal complement. Then, it is easily seen from (2.7) that

$$(2.12) \quad \begin{aligned} \mu(E, Z) &= (Y, E)\mu(Y, Z) + (Z, E)\mu(Y, Y) \\ &\quad + (JY, E)\mu(Y, JZ) + (JZ, E)\mu(Y, JY), \end{aligned}$$

$$(2.13) \quad \begin{aligned} \mu(E, JZ) &= (Y, E)\mu(JY, Z) + (Z, E)\mu(JY, Y) \\ &\quad + (JY, E)\mu(JY, JZ) + (JZ, E)\mu(JY, JY), \end{aligned}$$

$$(2.14) \quad \begin{aligned} \mu(E, Y) &= (Y, E)\mu(Z, Z) + (Z, E)\mu(Z, Y) \\ &\quad + (JY, E)\mu(Z, JZ) + (JZ, E)\mu(Z, JY), \end{aligned}$$

$$(2.15) \quad \begin{aligned} \mu(E, JY) &= (Y, E)\mu(JZ, Z) + (Z, E)\mu(JZ, Y) \\ &\quad + (JY, E)\mu(JZ, JZ) + (JZ, E)\mu(JZ, JY), \end{aligned}$$

$$(2.16) \quad \begin{aligned} &\mu(E, F) - \mu(F, E) \\ &= -\mu(E, Y)(F, Y) + \mu(F, Y)(E, Y) \\ &\quad + \mu(E, JY)(JF, Y) - \mu(F, JY)(JE, Y), \end{aligned}$$

$$(2.17) \quad \begin{aligned} &\mu(E, F) - \mu(F, E) \\ &= -\mu(E, Z)(F, Z) + \mu(F, Z)(E, Z) \\ &\quad + \mu(E, JZ)(JF, Z) - \mu(F, JZ)(JE, Z) \end{aligned}$$

hold for any $E, F \in TM$. By virtue of (2.14), (2.15) and (2.16), we get

$$(2.18) \quad \begin{aligned} &\mu(E, F) - \mu(F, E) \\ &+ \{(Z, E)(Y, F) - (Z, F)(Y, E)\}\mu(Z, Y) \\ &+ \{(JY, E)(Y, F) - (JY, F)(Y, E)\}\mu(Z, JZ) \\ &- \{(JY, E)(Y, F) - (JY, F)(Y, E)\}\mu(JZ, Z) \\ &+ \{(JZ, E)(Y, F) - (JZ, F)(Y, E)\}\mu(Z, JY) \\ &+ \{(Z, E)(JY, F) - (Z, F)(JY, E)\}\mu(JZ, Y) \\ &+ \{(JZ, E)(JY, F) - (JZ, F)(JY, E)\}\mu(JZ, JY) = 0 \end{aligned}$$

for any $E, F \in TM$. Also, from (2.12), (2.13) and (2.17), we find

$$\begin{aligned}
 (2.19) \quad & \mu(E, F) - \mu(F, E) \\
 & + \{(Z, F)(Y, E) - (Z, E)(Y, F)\}\mu(Y, Z) \\
 & + \{(JY, E)(Z, F) - (JY, F)(Z, E)\}\mu(Y, JZ) \\
 & + \{(JZ, E)(Z, F) - (JZ, F)(Z, E)\}\mu(Y, JY) \\
 & - \{(JZ, E)(Z, F) - (JZ, F)(Z, E)\}\mu(JY, Y) \\
 & + \{(Y, E)(JZ, F) - (Y, F)(JZ, E)\}\mu(JY, Z) \\
 & + \{(JY, E)(JZ, F) - (JY, F)(JZ, E)\}\mu(JY, JZ) = 0
 \end{aligned}$$

for any $E, F \in TM$. It follows from (2.18) and (2.19) that

$$(2.20) \quad \begin{cases} \mu(Z, Y) + \mu(Y, Z) = 0, & \mu(Z, JZ) - \mu(JZ, Z) = 0, \\ \mu(Z, JY) + \mu(JY, Z) = 0, & \mu(JZ, Y) + \mu(Y, JZ) = 0, \\ \mu(JZ, JY) + \mu(JY, JZ) = 0, & \mu(Y, JY) - \mu(JY, Y) = 0. \end{cases}$$

From (2.18), (2.19) and (2.20), we have

$$(2.21) \quad \begin{cases} \mu(Z, Y) = \mu(Y, Z) = \mu(Z, JZ) = \mu(JZ, Z) = 0 \\ \mu(Z, JY) = \mu(JY, Z) = \mu(JZ, Y) = \mu(Y, JZ) = 0 \\ \mu(JZ, JY) = \mu(JY, JZ) = \mu(Y, JY) = \mu(JY, Y) = 0. \end{cases}$$

Hence, by means of (2.18) and (2.21), we get (2.9). Moreover, it follows from (2.3), (2.8) and (2.9) that (2.10) and (2.11) hold. \square

Since the first Bianchi equation of R^0 holds, from Lemma 2.1, we conclude that

$$(2.22) \quad R^0(E, F)J = JR^0(E, F) \quad \text{and} \quad R^0(JE, JF) = R^0(E, F)$$

for any $E, F \in TM$. Moreover, making use of (2.22), we find

$$\begin{aligned}
 (2.23) \quad & Ric^0(JE, JF) = Ric^0(E, F) = Ric^0(F, E) \text{ and} \\
 & Ric^0(E, F) = \frac{1}{2}(\text{Trace of } JR^0(E, JF))
 \end{aligned}$$

where $Ric^0(E, F) := \sum_{i=1}^{2m} (R^0(e_i, E)F, e_i)$ and $(\text{Trace of } JR^0(E, JF)) := \sum_{i=1}^{2m} (JR^0(E, JF)e_i, e_i)$ for an orthonormal frame $\{e_1, \dots, e_{2m}\}$ of TM .

Hereafter, in this section, we assume that ξ is everywhere non-zero and $m \geq 3$. Next we calculate the difference between the Ricci tensors. It is clear from (2.1) and (2.9) that

$$\begin{aligned}
 (2.24) \quad Ric^0(E, F) - Ric(E, F) &= \sum_{i=1}^{2m} (R^0(e_i, E)F, e_i) - \sum_{i=1}^{2m} (R(e_i, E)F, e_i) \\
 &= \sum_{i=1}^{2m} (R^0(E, e_i)e_i, F) - \sum_{i=1}^{2m} (R(E, e_i)e_i, F) \\
 &= \sum_{i=1}^{2m} \mu(e_i, e_i)(E, F) + \mu(E, F) + \mu(JE, JF) \\
 &\quad - \sum_{i=1}^{2m} \mu(e_i, Je_i)(JE, F) - \sum_{i=1}^{2m} \nu(E, e_i, e_i)(\xi, F) \\
 &\quad + \sum_{i=1}^{2m} \nu(E, e_i, Je_i)(J\xi, F)
 \end{aligned}$$

for any $E, F \in TM$. Since (2.23) holds, subtracting (2.24) from the equation obtained by changing E (resp. F) into JE (resp. JF) in (2.24), it follows that

$$\begin{aligned}
 (2.25) \quad &\sum_{i=1}^{2m} \nu(E, e_i, e_i)(\xi, F) \\
 &= \sum_{i=1}^{2m} \nu(E, e_i, Je_i)(J\xi, F) + \sum_{i=1}^{2m} \nu(JE, e_i, e_i)(\xi, JF) \\
 &\quad - \sum_{i=1}^{2m} \nu(JE, e_i, Je_i)(J\xi, JF).
 \end{aligned}$$

If we put $F = \xi$ in (2.25), then

$$(2.26) \quad \sum_{i=1}^{2m} \nu(E, e_i, e_i) = - \sum_{i=1}^{2m} \nu(JE, e_i, Je_i)$$

holds for any $E \in TM$. If we subtract (2.24) from the equation obtained by interchanging E and F in (2.24), then we obtain

$$(2.27) \quad 2 \sum_{i=1}^{2m} \mu(e_i, Je_i)(JE, F)$$

$$\begin{aligned}
& + \sum_{i=1}^{2m} \nu(E, e_i, e_i)(\xi, F) - \sum_{i=1}^{2m} \nu(F, e_i, e_i)(\xi, E) \\
& = \sum_{i=1}^{2m} \nu(E, e_i, Je_i)(J\xi, F) - \sum_{i=1}^{2m} \nu(F, e_i, Je_i)(J\xi, E).
\end{aligned}$$

Putting $F = \xi$ in (2.27), we get

$$\begin{aligned}
(2.28) \quad & 2 \sum_{i=1}^{2m} \mu(e_i, Je_i)(JE, \xi) \\
& + \sum_{i=1}^{2m} \nu(E, e_i, e_i)|\xi|^2 - \sum_{i=1}^{2m} \nu(\xi, e_i, e_i)(\xi, E) \\
& = - \sum_{i=1}^{2m} \nu(\xi, e_i, Je_i)(J\xi, E).
\end{aligned}$$

If we replace e_i by Je_i in (2.28) and use (2.11), then we have

$$\begin{aligned}
(2.29) \quad & - 2 \sum_{i=1}^{2m} \mu(e_i, Je_i)(JE, \xi) \\
& + \sum_{i=1}^{2m} \nu(E, Je_i, Je_i)|\xi|^2 - \sum_{i=1}^{2m} \nu(\xi, Je_i, Je_i)(\xi, E) \\
& = \sum_{i=1}^{2m} \nu(\xi, e_i, Je_i)(J\xi, E).
\end{aligned}$$

Since

$$\sum_{i=1}^{2m} \nu(E, e_i, e_i)|\xi|^2 = \sum_{i=1}^{2m} \nu(E, Je_i, Je_i)|\xi|^2$$

and

$$\sum_{i=1}^{2m} \nu(\xi, e_i, e_i)(\xi, E) = \sum_{i=1}^{2m} \nu(\xi, Je_i, Je_i)(\xi, E)$$

hold, from (2.28) and (2.29), we find

$$(2.30) \quad 2 \sum_{i=1}^{2m} \mu(e_i, Je_i) = \sum_{i=1}^{2m} \nu(\xi, e_i, Je_i),$$

$$(2.31) \quad \sum_{i=1}^{2m} \nu(E, e_i, e_i) = \lambda(\xi, E)$$

for any $E \in TM$, where we put $|\xi|^2\lambda = \sum_{i=1}^{2m} \nu(\xi, e_i, e_i)$. It follows from (2.26), (2.30) and (2.31) that

$$(2.32) \quad \sum_{i=1}^{2m} \mu(e_i, Je_i) = 0$$

and

$$(2.33) \quad \sum_{i=1}^{2m} \nu(E, e_i, Je_i) = -\lambda(J\xi, E)$$

hold for any $E \in TM$. Using (2.1), (2.9) and (2.23), we get

$$(2.34) \quad \begin{aligned} Ric^0(E, F) - Ric(E, F) &= - \sum_{i=1}^{2m} \frac{1}{2} (R^0(E, JF)e_i, Je_i) \\ &\quad + \sum_{i=1}^{2m} \frac{1}{2} (R(E, JF)e_i, Je_i) \\ &= (m+1)(\mu(E, F) + \mu(JE, JF)) - \nu(E, JF, J\xi). \end{aligned}$$

Also making use of (2.1) and (2.9), we obtain

$$(2.35) \quad \begin{aligned} Ric^0(E, F) - Ric(E, F) &= \sum_{i=1}^{2m} (R^0(e_i, E)F, e_i) - \sum_{i=1}^{2m} (R(e_i, E)F, e_i) \\ &= 2m\mu(E, F) + 2\mu(JE, JF) - \nu(\xi, E, F) \\ &\quad + \nu(J\xi, E, JF). \end{aligned}$$

If we subtract (2.34) from (2.35) and use (2.3) and (2.10), then we have

$$(m-1)\{\mu(E, F) - \mu(JE, JF)\} - \nu(\xi, E, F) - \nu(JF, J\xi, E) = 0,$$

which yields that

$$(2.36) \quad \mu(E, \xi) = \mu(JE, J\xi)$$

for any $E \in TM$. Hence, from (2.7), we get for $E, F, G, H \in TM$

$$(2.37) \quad \begin{aligned} 2(m-1)\mu(E, F) &= \{2(m-1)\bar{a} + (\lambda + \epsilon)|\xi|^2\}(E, F) \\ &\quad - (\lambda + \epsilon)\left\{(E, \xi)(F, \xi) + (E, J\xi)(F, J\xi)\right\}, \end{aligned}$$

$$\begin{aligned}
(2.38) \quad & 2(m-1)|\xi|^2\nu(E, F, G) \\
& = -(\lambda + \epsilon)|\xi|^2 \left\{ (E, JG)(F, J\xi) - (F, JG)(E, J\xi) \right. \\
& \quad \left. + 2(J\xi, G)(JF, E) + (G, E)(F, \xi) - (G, F)(E, \xi) \right\} \\
& \quad - 2\{2\lambda + (m+1)\epsilon\}(J\xi, G) \left\{ (J\xi, F)(\xi, E) \right. \\
& \quad \left. - (J\xi, E)(\xi, F) \right\},
\end{aligned}$$

where

$$\bar{a} := \frac{1}{|\xi|^2}\mu(\xi, \xi), \quad \lambda := \frac{1}{|\xi|^2} \sum_{i=1}^{2m} \nu(\xi, e_i, e_i), \quad \epsilon := -\frac{1}{|\xi|^4} \nu(\xi, J\xi, J\xi).$$

Therefore we get the following theorem.

Theorem 2.3. *Suppose that M is a Kählerian manifold with the complex structure J , $\dim M \geq 6$ and there exists an everywhere non-zero Kählerian torse-forming vector field ξ . If the curvature tensor R^0 satisfies (2.6), then we have*

$$\begin{aligned}
(2.39) \quad & (R^0(E, F)G, H) - (R(E, F)G, H) \\
& = -\left\{ \bar{a} + \frac{\lambda + \epsilon}{2(m-1)}|\xi|^2 \right\} \left\{ (E, G)(F, H) - (F, G)(E, H) \right. \\
& \quad \left. - (E, JG)(JF, H) + (F, JG)(JE, H) + 2(JE, F)(JG, H) \right\} \\
& \quad + \frac{\lambda + \epsilon}{2(m-1)} \left[\left\{ (E, JG)(J\xi, F) - (F, JG)(J\xi, E) \right. \right. \\
& \quad \left. \left. + 2(E, JF)(J\xi, G) - (F, G)(\xi, E) + (E, G)(\xi, F) \right\}(\xi, H) \right. \\
& \quad \left. - \left\{ (E, JG)(\xi, F) - (F, JG)(\xi, E) + 2(E, JF)(\xi, G) \right. \right. \\
& \quad \left. \left. + (F, G)(J\xi, E) - (E, G)(J\xi, F) \right\}(J\xi, H) + (\xi, G) \left\{ (\xi, E)(F, H) \right. \right. \\
& \quad \left. \left. - (\xi, F)(E, H) - (J\xi, E)(JF, H) + (J\xi, F)(JE, H) \right\} \right. \\
& \quad \left. + (J\xi, G) \left\{ (J\xi, E)(F, H) - (J\xi, F)(E, H) + (\xi, E)(JF, H) \right. \right. \\
& \quad \left. \left. - (\xi, F)(JE, H) \right\} - 2(JG, H) \left\{ (J\xi, E)(\xi, F) - (J\xi, F)(\xi, E) \right\} \right] \\
& \quad + \frac{2\lambda + (m+1)\epsilon}{(m-1)|\xi|^2} \left\{ (\xi, E)(J\xi, F) - (\xi, F)(J\xi, E) \right\} \left\{ (J\xi, G)(\xi, H) \right. \\
& \quad \left. - (J\xi, H)(\xi, G) \right\}
\end{aligned}$$

for $E, F, G, H \in TM$.

From Theorem 2.3, we get

$$\begin{aligned}
 (2.40) \quad Ric^0(E, F) - Ric(E, F) &= 2(m+1)\bar{a}(E, F) + \frac{m}{m-1}(\lambda + \epsilon)|\xi|^2(E, F) \\
 &\quad - \frac{m\lambda + \epsilon}{m-1}\{(\xi, E)(\xi, F) + (J\xi, E)(J\xi, F)\}, \\
 (2.41) \quad r^0 - r &= 4m(m+1)\bar{a} + \{2m\lambda + 2(m+1)\epsilon\}|\xi|^2
 \end{aligned}$$

where $r^0 := \sum_{i=1}^{2m} Ric^0(e_i, e_i)$.

For the Levi-Civita connection ∇ , the Bochner curvature tensor B [2] is defined by

$$\begin{aligned}
 (B(E, F)G, H) &:= (R(E, F)G, H) + \frac{1}{2m+4}\left\{ (E, G)Ric(F, H) \right. \\
 &\quad - (F, G)Ric(E, H) + (F, H)Ric(E, G) - (E, H)Ric(F, G) \\
 &\quad + (JE, G)Ric(JF, H) - (JF, G)Ric(JE, H) + (JF, H)Ric(JE, G) \\
 &\quad \left. - (JE, H)Ric(JF, G) + 2(JE, F)Ric(JG, H) + 2(JG, H)Ric(JE, F) \right\} \\
 &\quad - \frac{r}{(2m+4)(2m+2)}\left\{ (E, G)(F, H) - (F, G)(E, H) + (JE, G)(JF, H) \right. \\
 &\quad \left. - (JF, G)(JE, H) + 2(JE, F)(JG, H) \right\}
 \end{aligned}$$

for any $E, F, G, H \in TM$. Similarly, we define the following tensor B^0 by

$$\begin{aligned}
 (B^0(E, F)G, H) &:= (R^0(E, F)G, H) + \frac{1}{2m+4}\left\{ (E, G)Ric^0(F, H) \right. \\
 &\quad - (F, G)Ric^0(E, H) + (F, H)Ric^0(E, G) - (E, H)Ric^0(F, G) \\
 &\quad + (JE, G)Ric^0(JF, H) - (JF, G)Ric^0(JE, H) + (JF, H)Ric^0(JE, G) \\
 &\quad \left. - (JE, H)Ric^0(JF, G) + 2(JE, F)Ric^0(JG, H) + 2(JG, H)Ric^0(JE, F) \right\} \\
 &\quad - \frac{r^0}{(2m+4)(2m+2)}\left\{ (E, G)(F, H) - (F, G)(E, H) + (JE, G)(JF, H) \right. \\
 &\quad \left. - (JF, G)(JE, H) + 2(JE, F)(JG, H) \right\}
 \end{aligned}$$

for any $E, F, G, H \in TM$. We call B^0 the Bochner curvature tensor with respect to the linear connection ∇^0 . Then, by virtue of (2.39), (2.40) and (2.41), we get the following Theorem.

Theorem 2.4. Suppose that M is a Kählerian manifold with the complex structure J , $\dim M \geq 6$ and there exists an everywhere non-zero Kählerian torse-forming vector field ξ . If the curvature tensor R^0 satisfies (2.6), then, for $E, F, G, H \in TM$,

$$(2.42) \quad (B^0(E, F)G, H) - (B(E, F)G, H) = \{2\lambda + (m+1)\epsilon\}C(E, F, G, H),$$

moreover, we have

$$B^0 = B \quad \text{if and only if} \quad 2\lambda + (m+1)\epsilon = 0,$$

where

$$\begin{aligned} C(E, F, G, H) := & -\frac{1}{2(m^2-1)(m+2)}|\xi|^2 \left\{ (E, G)(F, H) - (F, G)(E, H) - (E, JG)(JF, H) \right. \\ & \left. + (F, JG)(JE, H) + 2(JE, F)(JG, H) \right\} \\ & + \frac{1}{2(m-1)(m+2)} \left[\left\{ (E, JG)(J\xi, F) - (F, JG)(J\xi, E) + 2(E, JF)(J\xi, G) \right. \right. \\ & - (F, G)(\xi, E) + (E, G)(\xi, F) \left. \right\} (\xi, H) - \left\{ (E, JG)(\xi, F) - (F, JG)(\xi, E) \right. \\ & + 2(E, JF)(\xi, G) + (F, G)(J\xi, E) - (E, G)(J\xi, F) \left. \right\} (J\xi, H) \\ & + (\xi, G) \left\{ (\xi, E)(F, H) - (\xi, F)(E, H) - (J\xi, E)(JF, H) + (J\xi, F)(JE, H) \right\} \\ & + (J\xi, G) \left\{ (J\xi, E)(F, H) - (J\xi, F)(E, H) + (\xi, E)(JF, H) - (\xi, F)(JE, H) \right\} \\ & \left. - 2(JG, H) \left\{ (J\xi, E)(\xi, F) - (J\xi, F)(\xi, E) \right\} \right] \\ & + \frac{1}{(m-1)|\xi|^2} \left\{ (\xi, E)(J\xi, F) - (\xi, F)(J\xi, E) \right\} \left\{ (J\xi, G)(\xi, H) - (J\xi, H)(\xi, G) \right\}. \end{aligned}$$

§3. Kählerian submersions and the Bochner curvature tensor

Let (M, g, J) be as in §2 and $(\overline{M}, \bar{g}, \bar{J})$ a real $2n$ -dimensional almost complex manifold with the almost complex structure \bar{J} and metric \bar{g} . For simplicity, we denote the metric \bar{g} by (\cdot, \cdot) . A smooth surjective mapping $\pi : M \rightarrow \overline{M}$ is called a *Riemannian submersion* [1] if π has maximal rank and $\pi_*|_{(\text{Ker } \pi_*)^\perp}$ is linear

isometry, where π_* is the derivative mapping of π . Vectors on M which are in the kernel of π_* are tangent to the fibers $\widehat{M}_p (= \pi^{-1}(p), p \in \overline{M})$. We call these *vertical* vectors. Vectors which are orthogonal to vertical distribution are said to be *horizontal*. We denote the vertical and horizontal distributions in the tangent bundle of the total space M by $\mathcal{V}(M)$ and $\mathcal{H}(M)$, respectively. Then TM has the orthogonal decomposition: $TM = \mathcal{V}(M) \oplus \mathcal{H}(M)$. The projection mappings are denoted by $\mathcal{V} : TM \rightarrow \mathcal{V}(M)$ and $\mathcal{H} : TM \rightarrow \mathcal{H}(M)$. Let E and F be arbitrary vector fields on M . The O'Neill configuration tensors [1] of the Riemannian submersion $\pi : M \rightarrow \overline{M}$ are given by

$$T_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \quad A_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F.$$

The properties of T and A are well-known, contained in O'Neill's original paper, and included here only for completeness.

Lemma 3.1 ([1]). *Let $\pi : M \rightarrow \overline{M}$ be a Riemannian submersion. Then*

- (a) *at any point $p \in M$, the linear operators T_E and A_E are skew-symmetric,*
- (b) *$T_E\{\mathcal{H}(M)\} \subset \mathcal{V}(M)$ and $T_E\{\mathcal{V}(M)\} \subset \mathcal{H}(M)$,*
- (c) *$A_E\{\mathcal{H}(M)\} \subset \mathcal{V}(M)$ and $A_E\{\mathcal{V}(M)\} \subset \mathcal{H}(M)$,*
- (d) *T is vertical and A is horizontal, i.e., $T_E = T_{\mathcal{V}E}$ and $A_E = A_{\mathcal{H}E}$,*
- (e) *$T_V W = T_W V$ for all $V, W \in \mathcal{V}(M)$,*
- (f) *$A_X Y = A_Y X$ for all $X, Y \in \mathcal{H}(M)$.*

A Riemannian submersion $\pi : M \rightarrow \overline{M}$ is said to be a *Kählerian submersion* if $\pi_* \circ J = \overline{J} \circ \pi_*$. B. Watson [4] proved that the vertical and horizontal distributions are J -invariant. Moreover he showed the following theorem.

Theorem 3.2 ([4]). *Let $\pi : M \rightarrow \overline{M}$ be a Kählerian submersion. Then the base space and each fiber are Kählerian manifolds, and the horizontal distribution is integrable.*

Let $\pi : M \rightarrow \overline{M}$ be a Kählerian submersion. Then, from Theorem 3.2, we find $A = 0$. Geometrical features of the fibers will be distinguished by a caret (^). We obtain

Lemma 3.3 ([1], [4]). *Let X, Y be horizontal vector fields and U, V vertical vector fields. Then*

$$\begin{aligned} \nabla_U V &= T_U V + \widehat{\nabla}_U V, \\ \nabla_U X &= \mathcal{H}\nabla_U X + T_U X, \\ \nabla_X U &= \mathcal{V}\nabla_X U, \\ \nabla_X Y &= \mathcal{H}\nabla_X Y, \end{aligned}$$

where $\widehat{\nabla}$ is the family of Levi-Civita connections on fibres.

For vertical vectors V_1, V_2, V_3, V_4 at $p \in M$, let $(\widehat{R}(V_1, V_2)V_3, V_4)$ be the curvature tensor of the fiber $\widehat{M}_{\pi(p)}$ at p . The horizontal lift of the curvature tensor \overline{R} of \overline{M} will also denoted by \overline{R} , that is, $\pi_*(\overline{R}(X, Y)Z) = \overline{R}(\pi_*X, \pi_*Y)\pi_*Z$ at each $p \in M$. Then we have the following lemma.

Lemma 3.4 ([1], [4]). *Let U, V, W, W' be vertical vector fields and X, Y, Z, Z' horizontal vector fields. Then*

$$(3.1) \quad (R(U, V)W, W') = (\widehat{R}(U, V)W, W') + (T_U W, T_V W') - (T_V W, T_U W'),$$

$$(3.2) \quad (R(U, V)W, X) = ((\nabla_U T)_V W, X) - ((\nabla_V T)_U W, X),$$

$$(3.3) \quad (R(X, U)Y, V) = ((\nabla_X T)_U Y, V) + (T_U X, T_V Y),$$

$$(3.4) \quad (R(U, V)X, Y) = (T_U X, T_V Y) - (T_V X, T_U Y),$$

$$(3.5) \quad (R(X, Y)Z, U) = 0,$$

$$(3.6) \quad (R(X, Y)Z, Z') = (\overline{R}(X, Y)Z, Z').$$

Let ξ be an everywhere non-zero Kählerian torse-forming vector field of M satisfying (1.1). We put

$$\xi^H := \mathcal{H}\xi, \quad \xi^V := \mathcal{V}\xi.$$

Then, by virtue of Lemma 3.3, the following identities hold:

$$(3.7) \quad \mathcal{H}\nabla_X \xi^H = aX + bJX + \alpha(X)\xi^H + \beta(X)J\xi^H,$$

$$(3.8) \quad \mathcal{V}\nabla_X \xi^V = \alpha(X)\xi^V + \beta(X)\widehat{J}\xi^V,$$

$$(3.9) \quad \mathcal{H}\nabla_U \xi^H + T_U \xi^V = \alpha(U)\xi^H + \beta(U)J\xi^H,$$

$$(3.10) \quad \widehat{\nabla}_U \xi^V + T_U \xi^H = aU + b\widehat{J}U + \alpha(U)\xi^V + \beta(U)\widehat{J}\xi^V,$$

where $X \in \mathcal{H}(M)$, $U \in \mathcal{V}(M)$ and \widehat{J} is the induced almost complex structure of each fiber. For $U, V, W, W' \in \mathcal{V}(M)$ and $X, Y, Z, Z' \in \mathcal{H}(M)$, from (2.39), (3.1), (3.5) and (3.6), we get

$$(3.11) \quad (\widehat{R}(U, V)W, W') + (T_U W, T_V W') - (T_V W, T_U W') - (R^0(U, V)W, W') \\ = \left\{ \bar{a} + \frac{\lambda + \epsilon}{2(m-1)} |\xi|^2 \right\} \left\{ (U, W)(V, W') - (V, W)(U, W') \right. \\ \left. - (U, \widehat{J}W)(\widehat{J}V, W') + (V, \widehat{J}W)(\widehat{J}U, W') + 2(\widehat{J}U, V)(\widehat{J}W, W') \right\}$$

$$\begin{aligned}
& -\frac{\lambda + \epsilon}{2(m-1)} \left[\left\{ (U, \widehat{J}W)(\widehat{J}\xi^V, V) - (V, \widehat{J}W)(\widehat{J}\xi^V, U) \right. \right. \\
& + 2(U, \widehat{J}V)(\widehat{J}\xi^V, W) - (V, W)(\xi^V, U) + (U, W)(\xi^V, V) \left. \right\} (\xi^V, W') \\
& - \left\{ (U, \widehat{J}W)(\xi^V, V) - (V, \widehat{J}W)(\xi^V, U) + 2(U, \widehat{J}V)(\xi^V, W) \right. \\
& + (V, W)(\widehat{J}\xi^V, U) - (U, W)(\widehat{J}\xi^V, V) \left. \right\} (\widehat{J}\xi^V, W') \\
& + (\xi^V, W) \left\{ (\xi^V, U)(V, W') - (\xi^V, V)(U, W') - (\widehat{J}\xi^V, U)(\widehat{J}V, W') \right. \\
& + (\widehat{J}\xi^V, V)(\widehat{J}U, W') \left. \right\} + (\widehat{J}\xi^V, W) \left\{ (\widehat{J}\xi^V, U)(V, W') \right. \\
& - (\widehat{J}\xi^V, V)(U, W') + (\xi^V, U)(\widehat{J}V, W') - (\xi^V, V)(\widehat{J}U, W') \left. \right\} \\
& \left. - 2(\widehat{J}W, W') \left\{ (\widehat{J}\xi^V, U)(\xi^V, V) - (\widehat{J}\xi^V, V)(\xi^V, U) \right\} \right] \\
& - \frac{2\lambda + (m+1)\epsilon}{(m-1)|\xi|^2} \left\{ (\xi^V, U)(\widehat{J}\xi^V, V) - (\xi^V, V)(\widehat{J}\xi^V, U) \right\} \left\{ \right. \\
& \left. (\widehat{J}\xi^V, W)(\xi^V, W') - (\widehat{J}\xi^V, W')(\xi^V, W) \right\},
\end{aligned}$$

$$\begin{aligned}
(3.12) \quad & 2(m-1)|\xi|^2(R^0(X, Y)Z, U) \\
& = (\lambda + \epsilon)|\xi|^2 \left[\left\{ (X, JZ)(J\xi^H, Y) - (Y, JZ)(J\xi^H, X) \right. \right. \\
& + 2(X, JY)(J\xi^H, Z) - (Y, Z)(\xi^H, X) + (X, Z)(\xi^H, Y) \left. \right\} (\xi^V, U) \\
& - \left\{ (X, JZ)(\xi^H, Y) - (Y, JZ)(\xi^H, X) + 2(X, JY)(\xi^H, Z) \right. \\
& + (Y, Z)(J\xi^H, X) - (X, Z)(J\xi^H, Y) \left. \right\} (\widehat{J}\xi^V, U) \left. \right] \\
& + \{4\lambda + 2(m+1)\epsilon\} \left\{ (\xi^H, X)(J\xi^H, Y) - (\xi^H, Y)(J\xi^H, X) \right. \\
& \left. \right\} \left\{ (J\xi^H, Z)(\xi^V, U) - (\widehat{J}\xi^V, U)(\xi^H, Z) \right\},
\end{aligned}$$

$$\begin{aligned}
(3.13) \quad & (\overline{R}(X, Y)Z, Z') - (R^0(X, Y)Z, Z') \\
& = \left\{ \bar{a} + \frac{\lambda + \epsilon}{2(m-1)}|\xi|^2 \right\} \left\{ (X, Z)(Y, Z') - (Y, Z)(X, Z') \right. \\
& \left. - (X, JZ)(JY, Z') + (Y, JZ)(JX, Z') + 2(JX, Y)(JZ, Z') \right\}
\end{aligned}$$

$$\begin{aligned}
& -\frac{\lambda + \epsilon}{2(m-1)} \left[\left\{ (X, JZ)(J\xi^H, Y) - (Y, JZ)(J\xi^H, X) \right. \right. \\
& + 2(X, JY)(J\xi^H, Z) - (Y, Z)(\xi^H, X) + (X, Z)(\xi^H, Y) \left. \right\} (\xi^H, Z') \\
& - \left\{ (X, JZ)(\xi^H, Y) - (Y, JZ)(\xi^H, X) + 2(X, JY)(\xi^H, Z) \right. \\
& + (Y, Z)(J\xi^H, X) - (X, Z)(J\xi^H, Y) \left. \right\} (J\xi^H, Z') \\
& + (\xi^H, Z) \left\{ (\xi^H, X)(Y, Z') - (\xi^H, Y)(X, Z') - (J\xi^H, X)(JY, Z') \right. \\
& + (J\xi^H, Y)(JX, Z') \left. \right\} + (J\xi^H, Z) \left\{ (J\xi^H, X)(Y, Z') \right. \\
& - (J\xi^H, Y)(X, Z') + (\xi^H, X)(JY, Z') - (\xi^H, Y)(JX, Z') \left. \right\} \\
& \left. - 2(JZ, Z') \left\{ (J\xi^H, X)(\xi^H, Y) - (J\xi^H, Y)(\xi^H, X) \right\} \right] \\
& - \frac{2\lambda + (m+1)\epsilon}{(m-1)|\xi|^2} \left\{ (\xi^H, X)(J\xi^H, Y) - (\xi^H, Y)(J\xi^H, X) \right\} \left\{ \right. \\
& \left. (J\xi^H, Z)(\xi^H, Z') - (J\xi^H, Z')(\xi^H, Z) \right\},
\end{aligned}$$

Let $\{X_1, \dots, X_{2n}\}$ be an orthonormal frame of $\mathcal{H}(M)$ and $\{V_1, \dots, V_{2s}\}$ an orthonormal frame of $\mathcal{V}(M)$ respectively. In [3], it is known that

$$(3.14) \quad \sum_{l=1}^{2n} ((\nabla_{X_l} T)_U V, X_l) = 0$$

for every $U, V \in \mathcal{V}(M)$. By virtue of (2.39), (3.3) and (3.14), we get the following lemma.

Lemma 3.5. *For $U, V \in \mathcal{V}(M)$ and $X, Y \in \mathcal{H}(M)$, we have*

$$\begin{aligned}
(3.15) \quad & |T|^2 \\
& = \sum_{l=1}^{2n} \sum_{\alpha=1}^{2s} (R^0(V_\alpha, X_l)X_l, V_\alpha) + 4ns \left\{ \bar{a} + \frac{\lambda + \epsilon}{2(m-1)} |\xi|^2 \right\} \\
& - \frac{\lambda + \epsilon}{m-1} \{ 2n |\xi^V|^2 + 2s |\xi^H|^2 \} + \frac{2\{2\lambda + (m+1)\epsilon\} |\xi^H|^2 |\xi^V|^2}{(m-1)|\xi|^2},
\end{aligned}$$

where $|T|^2 := \sum_{l=1}^{2n} \sum_{\alpha=1}^{2s} (T_{V_\alpha} X_l, T_{V_\alpha} X_l)$.

We deal with the case where the Bochner curvature tensor B^0 with respect to the linear connection ∇^0 vanishes, namely

$$(3.16) \quad (R^0(E, F)G, H)$$

$$\begin{aligned}
& + \frac{1}{2m+4} \left\{ (E, G)Ric^0(F, H) - (F, G)Ric^0(E, H) \right. \\
& + (F, H)Ric^0(E, G) - (E, H)Ric^0(F, G) \\
& + (JE, G)Ric^0(JF, H) - (JF, G)Ric^0(JE, H) \\
& + (JF, H)Ric^0(JE, G) - (JE, H)Ric^0(JF, G) \\
& + 2(JE, F)Ric^0(JG, H) + 2(JG, H)Ric^0(JE, F) \left. \right\} \\
& - \frac{r^0}{(2m+4)(2m+2)} \left\{ (E, G)(F, H) - (F, G)(E, H) \right. \\
& + (JE, G)(JF, H) - (JF, G)(JE, H) \\
& + 2(JE, F)(JG, H) \left. \right\} = 0
\end{aligned}$$

for any $E, F, G, H \in TM$. Then we obtain the following lemma.

Lemma 3.6. *Let $\dim \bar{M} = 2n \geq 4$. If B^0 vanishes, then, for every $p \in M$,*

$$2\lambda(p) + (m+1)\epsilon(p) = 0, \quad \xi_p^H = 0 \quad \text{or} \quad \xi_p^V = 0.$$

Proof. If we substitute (3.16) into (3.12), then we get

$$\begin{aligned}
(3.17) \quad & - \frac{2(m-1)|\xi|^2}{2m+4} \left\{ (X, Z)Ric^0(Y, U) - (Y, Z)Ric^0(X, U) \right. \\
& + (JX, Z)Ric^0(JY, U) - (JY, Z)Ric^0(JX, U) \\
& + 2(JX, Y)Ric^0(JZ, U) = (\lambda + \epsilon)|\xi|^2 \left[\left\{ \right. \right. \\
& (X, JZ)(J\xi^H, Y) - (Y, JZ)(J\xi^H, X) + 2(X, JY)(J\xi^H, Z) \\
& - (Y, Z)(\xi^H, X) + (X, Z)(\xi^H, Y) \left. \right\} (\xi^V, U) \\
& - \left\{ (X, JZ)(\xi^H, Y) - (Y, JZ)(\xi^H, X) + 2(X, JY)(\xi^H, Z) \right. \\
& + (Y, Z)(J\xi^H, X) - (X, Z)(J\xi^H, Y) \left. \right\} (\hat{J}\xi^V, U) \left. \right] \\
& + \{4\lambda + 2(m+1)\epsilon\} \left\{ (\xi^H, X)(J\xi^H, Y) \right. \\
& \left. - (\xi^H, Y)(J\xi^H, X) \right\} \left\{ J\xi^H, Z \right\} (\xi^V, U) - (\hat{J}\xi^V, U)(\xi^H, Z) \left. \right\}.
\end{aligned}$$

We put $Y = Z = X_l$ in (3.17) and take the summation over $l = 1, \dots, 2n$, then we have

$$\frac{2(m-1)(n+1)|\xi|^2}{m+2} Ric^0(X, U)$$

$$= 2 \left\{ -(\lambda + \epsilon)(n+1)|\xi|^2 + \{2\lambda + (m+1)\epsilon\}|\xi^H|^2 \right\} \left\{ (X, \xi^H)(U, \xi^V) + (X, J\xi^H)(U, \widehat{J}\xi^V) \right\}.$$

In the above equation, putting $X = \xi^H$ and $U = \xi^V$, we obtain

$$(3.18) \quad \frac{2(m-1)(n+1)|\xi|^2}{m+2} Ric^0(\xi^H, \xi^V) \\ = 2 \left\{ -(\lambda + \epsilon)(n+1)|\xi|^2 + \{2\lambda + (m+1)\epsilon\}|\xi^H|^2 \right\} |\xi^H|^2 |\xi^V|^2.$$

Also, if we put $U = \xi^V$ and $X = \xi^H$ in (3.17), then we obtain

$$\begin{aligned} & \frac{-2(m-1)|\xi|^2}{2m+4} \left\{ (\xi^H, Z) Ric^0(Y, \xi^V) - (Y, Z) Ric^0(\xi^H, \xi^V) + (J\xi^H, Z) Ric^0(JY, \xi^V) \right. \\ & \quad \left. - (JY, Z) Ric^0(J\xi^H, \xi^V) + 2(J\xi^H, Y) Ric^0(JZ, \xi^V) \right\} \\ & = |\xi^V|^2 \left[(\lambda + \epsilon)|\xi|^2 \left\{ (\xi^H, Z)(\xi^H, Y) - (Y, Z)|\xi^H|^2 \right\} \right. \\ & \quad \left. + \left\{ 2\{2\lambda + (m+1)\epsilon\}|\xi^H|^2 - 3(\lambda + \epsilon)|\xi|^2 \right\} (J\xi^H, Y)(J\xi^H, Z) \right]. \end{aligned}$$

Substituting $Y = Z = J\xi^H$ into the above equation, we have

$$(3.19) \quad \frac{2(m-1)|\xi|^2}{m+2} Ric^0(\xi^H, \xi^V) |\xi^H|^2 \\ = |\xi^V|^2 |\xi^H|^4 \left\{ \{2\lambda + (m+1)\epsilon\} |\xi^H|^2 - 2(\lambda + \epsilon) |\xi|^2 \right\}.$$

It follows from (3.18) and (3.19) that

$$\{2\lambda + (m+1)\epsilon\} |\xi^H|^6 |\xi^V|^2 = 0.$$

□

We put

$$\begin{aligned} U_1 &:= \{p \in M | \xi_p^H \neq 0\} \cap \{p \in M | \xi_p^V \neq 0\}, \\ U_2 &:= \{p \in M | \xi_p^H = 0\}, \\ U_3 &:= \{p \in M | \xi_p^V = 0\}. \end{aligned}$$

Let U_2° (resp. U_3°) be the set of all interior points of U_2 (resp. U_3). We prove

Lemma 3.7. *If ξ is everywhere non-zero proper, then $U_2^\circ = U_3^\circ = \emptyset$.*

Proof. Suppose that $U_2^\circ \neq \emptyset$. From (3.7), we have $aX + bJX = 0$ for every $X \in \mathcal{H}(U_2^\circ)$. Hence we get $a = b = 0$ on U_2 . This contradicts the fact that ξ is proper. Therefore, we see that $U_2^\circ = \emptyset$.

Next suppose that $U_3^\circ \neq \emptyset$. From (3.10), we obtain $T_U \xi^H = aU + b\hat{J}U$ for $U \in \mathcal{V}(U_3^\circ)$. Moreover we find $(\xi^H, T_U U) = -(T_U \xi^H, U) = -a(U, U)$ for every $U \in \mathcal{V}(U_3^\circ)$. Since each fiber is minimal, we see that $a = 0$ on U_3 . Also since $JT_U U = T_U \hat{J}U$, we have $(\xi^H, JT_U U) = (\xi^H, T_U \hat{J}U) = -(T_U \xi^H, \hat{J}U) = -b(U, U)$. Hence we conclude that $b = 0$ on U_3 . This contradicts the fact that ξ is proper. Therefore, we see that $U_3^\circ = \emptyset$. \square

From Theorem 2.4, Lemma 3.6 and Lemma 3.7, we have the following theorem.

Theorem 3.8. *Suppose that $\pi : M \rightarrow \bar{M}$ is a Kählerian submersion, $\dim M \geq 6$, $\dim \bar{M} \geq 4$ and there exists an everywhere non-zero proper Kählerian torsion-forming vector field ξ . If the Bochner curvature tensor B^0 with respect to ∇^0 vanishes, then the Bochner curvature tensor B of M vanishes.*

By virtue of Theorem 3.8 and (3.13), we obtain

Corollary 3.9. *Let π, ξ, ρ, f be as in Theorem 3.8. If the Bochner curvature tensor B^0 with respect to ∇^0 vanishes, then the Bochner curvature tensor of \bar{M} vanishes.*

Next we deal with the case where the curvature tensor R^0 of ∇^0 satisfies

$$\begin{aligned}
 (3.20) \quad & (R^0(E, F)G, H) \\
 &= -\frac{r^0}{4m(m+1)} \{ (E, G)(F, H) - (F, G)(E, H) \\
 &\quad - (E, JG)(JF, H) + (F, JG)(JE, H) \\
 &\quad + 2(JE, F)(JG, H) \}
 \end{aligned}$$

for any $E, F, G, H \in TM$. We can easily see that B^0 vanishes, because R^0 satisfies (3.20). By the quite same method as the proof of Lemma 3.6, we get the following lemma.

Lemma 3.10. *Let $\dim \bar{M} = 2n \geq 4$. If R^0 is a tensor satisfies (3.20), then, for every $p \in M$,*

$$\lambda(p) = \epsilon(p) = 0, \quad \xi_p^H = 0 \quad \text{or} \quad \xi_p^V = 0.$$

If the curvature tensor R of the Levi-Civita connection ∇ satisfies (3.20), then M is said to be a space of constant holomorphic sectional curvature. From Lemma 3.5, Lemma 3.7 and Lemma 3.10, we have the following theorem.

Theorem 3.11. *Suppose that $\pi : M \rightarrow \overline{M}$ is a Kählerian submersion, $\dim M \geq 6$, $\dim \overline{M} \geq 4$ and there exists an everywhere non-zero proper Kählerian torse-forming vector field ξ . If the curvature tensor R^0 satisfies (3.20), then M is of non-positive constant holomorphic sectional curvature.*

By virtue of Theorem 3.11, (3.11) and (3.13), we obtain

Corollary 3.12. *Let π, ξ, ρ, f be as in Theorem 3.11, If the curvature tensor R^0 satisfies (3.20), then \overline{M} is of non-positive constant holomorphic sectional curvature and each fiber is an Einstein manifold.*

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